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The London equations for superconductors in a gravitational field

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Abstract. We derive the modified London equations for a superconductor in a gravitational field, write these equations in an elegant covariant form, and show that these equations are consistent with the modified fluxoid quantisation condition in a gravitational field found by DeWitt.

1. Introduction

Fluxoid quantisation in superconductors was first suggested by London (1950) and later considered by Onsager (1954, 1961) and by Byers and Yang (1961). It was first observed experimentally by Deaver and Fairbanks (1961) and by Doll and Näbauer (1961). Fluxoid quantisation in superconductors in the presence of a magnetic-type gravitational field was first considered by DeWitt (1966), who showed that in a gravitational field $\boldsymbol{g}$ set up by a changing magnetic-type gravitational field $\boldsymbol{P}$, this fluxoid quantisation condition must be modified. We will briefly rederive the modified flux quantisation condition of DeWitt using simple physical arguments and a description of general relativity which looks similar to Maxwell's equations. We will find that, in the presence of a magnetic-type gravitational field, magnetic fields $\boldsymbol{B}$ and supercurrents $\boldsymbol{J}_s$ exist deep within the superconductor and that the phenomenological London equations must also be modified. We derive these modified London equations and show that they can be written in an elegant covariant form. These equations, which are our primary result, can be applied to various experimental situations even by someone not familiar with general relativity.

Gravitational effects in superconductors serve as a potential test of general relativity. Papini (1966, 1967) has shown, however, that such effects are quite small if laboratory sources or the rotating earth supply the $\boldsymbol{P}$ field. The effects could be quite large near a neutron star.

In § 2 we will write down the weak-field gravitational field equations in terms of $\boldsymbol{g}$ and $\boldsymbol{P}$ in a form very similar to Maxwell's equations. In § 3 we derive the modified quantisation condition and in § 4 we derive the London equations in covariant form.

2. Gravitational field equations in vector form

Forward (1961) first wrote down the gravitational field equations of general relativity in a vector form similar to Maxwell's equations. He called the gravitational field $\boldsymbol{P}$, which arises from moving masses and which is therefore analogous to the magnetic field in electricity and magnetism, the 'protational' field, and we shall use this nomenclature. It is also referred to as a magnetic-type gravitational field. A more modern version of the equations for weak fields and low velocities written in terms of the PPN parameters has been given by Braginsky *et al* (1977), and we use their equations in the following. These equations are very convenient for doing practical calculations. For simplicity, we shall choose all of the PPN parameters to take on the values for general relativity itself. The more general case is easily handled. For general relativity, we have

$$\nabla \cdot \boldsymbol{g} = -4\pi G\rho_0\left(1+2\frac{v^2}{c^2}+\frac{I}{c^2}+\frac{3p}{\rho_0 c^2}\right)+\frac{3}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{1}$$

$$\nabla \times \boldsymbol{g} = -(1/c)\partial \boldsymbol{P}/\partial t \tag{2}$$

$$\nabla \cdot \boldsymbol{P} = 0 \tag{3}$$

$$\nabla \times \boldsymbol{P} = -16\pi G\frac{\rho_0 \boldsymbol{v}}{c}+\frac{4}{c}\frac{\partial \boldsymbol{g}}{\partial t} \tag{4}$$

where ρ_0 is the density of rest mass in the local rest frame of the matter, $\boldsymbol{v}$ is the coordinate velocity of the rest mass relative to the PPN frame, I is the specific internal energy, p is pressure and ϕ is a gravitational potential. We will be most interested in (4), which tells how moving matter produces a protational field $\boldsymbol{P}$, and in (2), which shows how a changing $\boldsymbol{P}$ field produces a $\boldsymbol{g}$ field. We clearly have a very direct correspondence between the usual gravitational acceleration field $\boldsymbol{g}$ and an electric field and between $\boldsymbol{P}$ and a magnetic field. Equation (2) can be written in the form

$$\oint \boldsymbol{g}\cdot \mathrm{d}\boldsymbol{l} = -\frac{1}{c}\frac{\partial}{\partial t}\Phi_P \tag{5}$$

where Φ_P is the flux of $\boldsymbol{P}$ through the loop defined by the integral on the left. The rotating earth, for example, will produce a $\boldsymbol{P}$ field according to (4) in the same way that a rotating charge produces a magnetic field.

3. Fluxoid quantisation in the presence of a protational field

We shall derive fluxoid quantisation in a manner similar to the treatment of Rose-Innes and Rhoderick (1969), but with the protational field included. Assume that a mass m with charge q is moving in a field-free region with velocity $\boldsymbol{v}_1$ and that a protational field $\boldsymbol{P}$ is applied at time $t=0$. While $\boldsymbol{P}$ is building up, there will be an induced $\boldsymbol{g}$ field given by (2). The momentum of the mass at time t will then be

$$m\boldsymbol{v}_2 = m\boldsymbol{v}_1 + \int_0^t m\boldsymbol{g}\,\mathrm{d}t. \tag{6}$$

Define a potential $\boldsymbol{Q}$ such that

$$\boldsymbol{P} = \mathrm{curl}\,\boldsymbol{Q}. \tag{7}$$

Then

$$\mathrm{curl}\,\boldsymbol{g} = -(1/c)\mathrm{d}(\mathrm{curl}\,\boldsymbol{Q})/\mathrm{d}t \tag{8}$$

and

$$g = -\frac{1}{c}\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t} + \nabla\chi \tag{9}$$

where χ is related to the usual gravitational scalar potential. This $\nabla\chi$ term in (9) will not contribute to (14) and (15) below since its curl is zero. Thus, for simplicity, we will omit it from the following discussion. Putting (9) into (6) then gives the momentum at time t as

$$m\boldsymbol{v}_2 = m\boldsymbol{v}_1 - (m/c)\boldsymbol{Q} \tag{10}$$

so that the momentum

$$\boldsymbol{\pi} \equiv m\boldsymbol{v} + (m/c)\boldsymbol{Q} \tag{11}$$

is conserved during application of the field and is the effective momentum when a $\boldsymbol{P}$ field is present. If we also include the effects of a changing magnetic field, we can generalise (11) to

$$\boldsymbol{\pi} \equiv m\boldsymbol{v} + q\boldsymbol{A} + (m/c)\boldsymbol{Q} \tag{12}$$

where $\boldsymbol{A}$ is the electromagnetic vector potential. For electron pairs, the appropriate momentum is $\boldsymbol{\pi}_{\mathrm{p}} = 2\boldsymbol{\pi}$ if m and q are the mass and charge of an electron in (12).

Now consider the phase of the electron-pair wavefunction in a superconductor. The phase difference between two points x and y of the superconductor is just

$$(\Delta\phi)_{xy} = 2\pi\int_x^y \frac{\hat{x}\cdot\mathrm{d}\boldsymbol{l}}{\lambda} = \frac{2\pi}{h}\int_x^y \boldsymbol{\pi}_{\mathrm{p}}\cdot\mathrm{d}\boldsymbol{l} \tag{13}$$

where λ is the wavelength of the electron pairs and $\boldsymbol{\pi}_{\mathrm{p}}$ their momentum defined above. For a closed path, this phase difference must be $2\pi n$, where n is an integer, so we have

$$2\pi n = \frac{4\pi m}{hn_{\mathrm{s}}e}\oint \boldsymbol{J}_{\mathrm{s}}\cdot\mathrm{d}\boldsymbol{l} + \frac{4\pi e}{h}\oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} + \frac{4\pi}{h}\frac{m}{c}\oint \boldsymbol{Q}\cdot\mathrm{d}\boldsymbol{l} \tag{14}$$

where m and e are now the mass and charge of an electron. $\boldsymbol{J}_{\mathrm{s}} \equiv en_{\mathrm{s}}\boldsymbol{v}$ is the supercurrent density and n_{s} is the density of superconducting electrons. Using Stoke's theorem, (7) and $\boldsymbol{B} = \mathrm{curl}\,\boldsymbol{A}$ allows us to write (14) as

$$n\Phi_0 = \frac{m}{n_{\mathrm{s}}e^2}\oint \boldsymbol{J}_{\mathrm{s}}\cdot\mathrm{d}\boldsymbol{l} + \Phi_B + \frac{m}{ec}\Phi_P \tag{15}$$

where $\Phi_0 \equiv h/2e$, Φ_B is the flux of the magnetic field $\boldsymbol{B}$ through the integration loop and Φ_P is the flux of the protational gravitational field $\boldsymbol{P}$ through the loop. Equation (15) is equivalent to the DeWitt (1966) quantisation condition for a superconductor in a protational gravitational field. We see that $\boldsymbol{P}$ and $\boldsymbol{B}$ play very similar roles.

Equation (15) has immediate consequences if $\Phi_P \neq 0$. Consider an integration path deep within a superconductor (not near the surface) and surrounding a normal region. The gravitational field $\boldsymbol{P}$ from an external source such as the earth penetrates the superconductor just as it does any other matter. Self-shielding due to mass currents in the superconductor is negligible. If we change the integration path an infinitesimal amount, Φ_P will change infinitesimally. If (15) is to be maintained for some value of n, Φ_B and/or the $\boldsymbol{J}_{\mathrm{s}}$ term must also change infinitesimally. Since Φ_B and $\boldsymbol{J}_{\mathrm{s}}$ are related, in general this means that both $\boldsymbol{B}$ and $\boldsymbol{J}_s$ must be non-zero deep within the superconductor when Φ_P is present. This of course implies that the Meissner effect is not exact when Φ_P is present and that the London equations must be modified.

If we consider a closed path deep within a superconductor with no normal regions present, we expect $n=0$ in (15). In that case, (15) can be written in differential form as

$$\frac{m_e}{n_s e^2}\operatorname{curl}\boldsymbol{J}_s+\boldsymbol{B}+\frac{m_e}{ec}\boldsymbol{P}=0. \tag{16}$$

Thus we expect (16) to replace the usual London equation. Also, for any perfect conductor in the presence of an electric field $\boldsymbol{E}$ and a gravitational acceleration field $\boldsymbol{g}$, we expect

$$\dot{\boldsymbol{J}}_s=\frac{n_s e^2}{m}\boldsymbol{E}+n_s e\boldsymbol{g} \tag{17}$$

as a further modification of the London equations. In the following section we shall write the London equations in a new covariant form and show that (16) and (17) are indeed consistent with this covariant formulation.

4. London equations in covariant form

We can write the usual London equations very succinctly as

$$F^{\mu\nu}_{|\nu}=\frac{s^\mu}{\sqrt{\varepsilon_0}} \tag{18}$$

and

$$\Pi_{\mu|\alpha}-\Pi_{\alpha|\mu}=0. \tag{19}$$

Here the $F^{\mu\nu}$ is the electromagnetic field tensor, $s^\mu=(\rho, j/c)$ is the source vector and $\Pi_\mu\equiv mv_\mu+eA_\mu$ is the momentum four-vector. A vertical bar represents a partial derivative. Equation (18) gives the Maxwell equations

$$\operatorname{curl}\boldsymbol{B}/\mu_0=\boldsymbol{j}+\dot{\boldsymbol{E}}\varepsilon_0 \tag{20}$$

and

$$\operatorname{div}\boldsymbol{E}=\rho/\varepsilon_0 \tag{21}$$

while (19) gives

$$(m/n_s e^2)\operatorname{curl}\boldsymbol{J}_s+\boldsymbol{B}=0 \tag{22}$$

for the (space, space) indices and

$$\dot{\boldsymbol{J}}_s=(n_s e^2/m)\boldsymbol{E}+n_s ec\nabla v_0 \tag{23}$$

for the (space, time) indices, using $\boldsymbol{J}_s\equiv n_s e\boldsymbol{v}$. v_0 is the zeroth component of the four-velocity so ∇v_0 just represents any other external force which can act on the electrons and can be deleted. Equations (20)–(23) give the complete set of equations for a superconductor. If now we put our superconductor in a gravitational field, (18) becomes

$$F^{\mu\nu}_{\|\nu} = s^{\mu}/\sqrt{\varepsilon_0} \tag{24}$$

where we now have a covariant derivative. Equation (19) has the same form in a gravitational field only now the four-momentum is

$$\Pi_{\mu} \equiv mv_{\mu} + eA_{\mu} + (m/c)Q_{\mu}, \tag{25}$$

in agreement with (12). Q_{μ} is a gravitational vector potential where $Q_{\mu} \equiv (-\psi, \boldsymbol{Q})$ and

$$\boldsymbol{g} \equiv -\nabla\psi - \frac{1}{c}\frac{\partial \boldsymbol{Q}}{\partial t} \tag{26}$$

$$\boldsymbol{P} \equiv \mathrm{curl}\, \boldsymbol{Q}. \tag{27}$$

The (space, space) part of (19) then gives (16) exactly as we want with an additional $n_s ec\nabla v_0$ term as above which can be deleted. The (space, time) part of (19) gives (17) again exactly as we would like. We can write out (24) as

$$F^{\mu\nu}_{|\nu} + F^{\mu\nu}\frac{(\sqrt{-g})_{|\nu}}{\sqrt{-g}} = \frac{s^{\mu}}{\sqrt{\varepsilon_0}}. \tag{28}$$

The $(\sqrt{-g})_{|\nu}$ term vanishes for the Schwarzschild metric in rectangular coordinates (in curvilinear coordinates this term is necessary, of course, but does not depend on the central mass). For the Kerr metric the leading non-trivial, non-zero term is of order m^2a^4/r^6 where m is the Schwarzschild mass parameter and a is the Kerr metric rotation parameter. All lower-order terms cancel out. This term is thus so small for most applications that it can be neglected in rectangular coordinates.

We conclude, then, that (24) and (19) are a covariant description of a superconductor in a gravitational field where the gravitational vector potential Q_{μ} must be added to the four-momentum as in (25). Equations (24), (19) and (25) reproduce (16) and (17) and, of course, are consistent with the quantisation condition (15). Thus we end up with an elegant, covariant, self-consistent description of a superconductor in a gravitational field.

References

Braginsky V B, Caves C M and Thorne K S 1977 *Phys. Rev.* D **15** 2047

Byers N and Yang C N 1961 *Phys. Rev. Lett.* **7** 46

Deaver B S and Fairbanks W M 1961 *Phys. Rev. Lett.* **7** 43

DeWitt B S 1966 *Phys. Rev. Lett.* **16** 1902

Doll R and Näbauer M 1961 *Phys. Rev. Lett.* **7** 51

Forward R L 1961 *Proc. Inst. Radio Engr.* **49** 892

London F 1950 *Superfluids* (New York: Wiley)

Onsager L 1954 *Proc. Int. Conf. Theoretical Physics, Kyoto and Tokyo, September 1953* (Tokyo: Science Council of Japan) p 935

——1961 *Phys. Rev. Lett.* **7** 50

Papini G 1966 *Nuovo Cimento* B **45** 66

——1967 *Nuovo Cimento Lett.* B **52** 136

Rose-Innes A C and Rhoderick E H 1969 *Introduction to Superconductivity* (Oxford: Pergamon)